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Defining relation for semi-invariants of three by three matrices triples

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Abstract

The single defining relation of the algebra of $SL_3 \times SL_3$-invariants of triples of 3×3 matrices is explicitly computed. Connections to some other prominent algebras of invariants are pointed out.

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1 Introduction

Denote by $M_n(K)$ the space of $n \times n$ matrices over an infinite field K. The direct product of two copies of the general linear group $G_n := GL_n(K) \times GL_n(K)$ acts linearly on $M_n(K)$: The group element (g, h) maps $A \in M_n(K)$ to gAh^{-1}. Take the direct sum $M_n(K)^m := \underbrace{M_n(K) \oplus \cdots \oplus M_n(K)}_{m}$ of m copies of this representation of G_n. The action of G_n induces an action on the algebra of polynomial functions $K[M_n(K)^m]$ in the usual way. Let $R_{n,m}(K)$ be the subalgebra of the invariants of the subgroup $SL_n(K) \times SL_n(K)$ of G_n. It is called also the algebra of semi-invariants of G_n on $M_n(K)^m$. The structure and minimal systems of generators of $R_{n,m}(K)$ are known in few cases only. Over a field of characteristic 0 or $p > 2$ the algebra $R_{2,m}(K)$ is minimally generated by the determinants $\det(A_r)$, $r = 1, \ldots, m$, the mixed discriminants $M(A_{r_1}, A_{r_2})$, $1 \leq r_1 < r_2 \leq m$, and the discriminants $D(A_{r_1}, A_{r_2}, A_{r_3}, A_{r_4})$, $1 \leq r_1 < r_2 < r_3 < r_4 \leq m$, [11] or [20, Theorem 11.47]. Here $M(A_1, A_2)$ is defined as the coefficient of $t_1 t_2$ in

$$\det(t_1 A_1 + t_2 A_2) = t_1^2 \det(A_1^2) + t_1 t_2 M(A_1, A_2) + t_2^2 \det(A_2),$$

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$$D(A_1,A_2,A_3,A_4) = \begin{vmatrix} a_{11}^{(1)} & a_{11}^{(2)} & a_{11}^{(3)} & a_{11}^{(4)} \\ a_{21}^{(1)} & a_{21}^{(2)} & a_{21}^{(3)} & a_{21}^{(4)} \\ a_{12}^{(1)} & a_{12}^{(2)} & a_{12}^{(3)} & a_{12}^{(4)} \\ a_{22}^{(1)} & a_{22}^{(2)} & a_{22}^{(3)} & a_{22}^{(4)} \end{vmatrix},$$

where $A_r = \left(a_{ij}^{(r)}\right)_{2\times 2}$, $r = 1,2,3,4$. For $m \le 4$ the generators $\det(A_r)$ and $M(A_{r_1}, A_{r_2})$ are algebraically independent and for $m = 4$ the algebra $R_{2,4}(K)$ is a free module over the polynomial subalgebra generated by them with basis $1, D(A_1,A_2,A_3,A_4)$. It is pointed out in [13], [20] that $R_{2,m}(K)$ can be interpreted as the ring of vector invariants of the special orthogonal group of degree 4. Therefore the relations among the generators can be deduced from classically known results, and even a Gröbner basis of the ideal of relations can be obtained from [13]. For $m = 2$ and any n the algebra $R_{n,2}$ is generated by the algebraically independent coefficients of $\det(t_1A_1 + t_2A_2)$, [17], see also [19].

Apart from the cases $n = 2$ for any m and $m = 2$ for any n, the only other case when a minimal system of generators of $R_{n,m}(K)$ is explicitly known is $n = m = 3$, and this algebra is the main object to study in the present paper. In the sequel we denote $R(K) := K[M_3(K)^3]^{SL_3\times SL_3}$, the algebra of $SL_3(K)\times SL_3(K)$-invariant polynomial functions on $M_3(K)^3$.

Define polynomial functions $f_{i,j,k}$ on $M_3(K)^3$ by the equality

$$\det(t_1A_1 + t_2A_2 + t_3A_3) = \sum_{i+j+k=3} t_1^i t_2^j t_3^k f_{i,j,k}(A_1,A_2,A_3)$$

for all $t_1,t_2,t_3 \in K$ and $A_1,A_2,A_3 \in M_3(K)$. Obviously the ten polynomials $f_{i,j,k}$ belong to $R(K)$. Furthermore, define h as the coefficient of $t_1^2t_2^2t_3^2$ in

$$\det\begin{pmatrix} t_2A_2 & t_1A_1 \\ t_1A_1 & t_3A_3 \end{pmatrix}$$

and define q as the coefficient of $t_1^2t_2t_3^2t_4t_5^2t_6$ in

$$\det\begin{pmatrix} 0 & t_1A_1 & t_2A_2 \\ t_4A_1 & 0 & t_3A_3 \\ t_5A_2 & t_6A_3 & 0 \end{pmatrix}.$$

Clearly h and q belong to $R(K)$. It is proved in [10] that h and the ten polynomials $f_{i,j,k}$ (where $i + j + k = 3$) constitute a homogeneous system of parameters in $R(K)$. Denote by $P(K)$ the subalgebra generated by these eleven algebraically independent elements. In the case when the characteristic of the base field K is zero, using a result of Teranishi [23] it was established in [10] that $R(K)$ is a free $P(K)$-module generated by 1 and q:

$$R(K) = P(K) \oplus P(K)q. \tag{1}$$

A similar description of $R(K)$ is stated by Mukai without proof in [20, Proposition 11.49]. It follows from (1) that q satisfies a monic quadratic relation with coefficients from $P(K)$. In the present paper we find the explicit form of this relation.

A crucial role in our considerations is played by the following right action of the general linear group $GL_3(K)$ on $M_3(K)^3$: For $g = (g_{ij})_{3\times 3} \in GL_3(K)$ and $(A_1, A_2, A_3) \in M_3(K)^3$ we have

$$(A_1, A_2, A_3) \cdot g := \left(\sum_{i=1}^{3} g_{i1} A_i, \sum_{i=1}^{2} g_{i2} A_i, \sum_{i=1}^{3} g_{i3} A_i \right).$$

This induces a left action of $GL_3(K)$ on the coordinate ring of $M_3(K)^3$: For a polynomial function f on $M_3(K)^3$ and $g \in GL_3(K)$, the function $g \cdot f$ maps $(A_1, A_2, A_3) \in M_3(K)^3$ to $f((A_1, A_2, A_3) \cdot g)$. Since this action of $GL_3(K)$ commutes with the action of G introduced above, $R(K)$ is a $GL_3(K)$-submodule of the coordinate ring of $M_3(K)^3$.

First in Section 2 we treat the case when K is the field $\mathbb{Q}$ of rational numbers. By the theory of polynomial representations of $GL_3(\mathbb{Q})$ one can read off from (1) that h and q can be replaced by H and Q that are *highest weight vectors* with respect to $GL_3(\mathbb{Q})$. In fact H and Q are invariants with respect to the subgroup $SL_3(\mathbb{Q})$ of $GL_3(\mathbb{Q})$ and they are uniquely determined up to non-zero scalar multiples. The relation among the new generators $Q, H, f_{i,j,k}$ takes place in the subalgebra of $SL_3(\mathbb{Q})$-invariants in $R(\mathbb{Q})$. This is a "small" subalgebra of $R(\mathbb{Q})$, and a "large" part of it can be identified with the algebra of $SL_3(\mathbb{C})$-invariants of ternary cubic forms, whose explicit generators S and T are known from a famous classical computation of Aronhold [3]. It is an easy matter to find H and Q explicitly, and then most of the computational difficulty in finding the relation among $Q, H, f_{i,j,k}$ is already contained in Aronhold's computation, so one gets easily the desired relation (cf. Theorem 1).

Rewriting the relation found in Section 2 in terms of our original generators q,h, $f_{i,j,k}$, we obtain a relation $A(q, h, f_1, \ldots, f_{10}) = 0$ with integer coefficients. This yields a uniform description for $R(K)$ in terms of a minimal generating system and the corresponding defining relations, valid over any infinite base field K and also for $K = \mathbb{Z}$, the ring of integers, see Theorem 3.

The results in Theorems 1 and 3 can be applied to recover in a transparent way known results in three other topics of independent interest. In Remark 2 we mention the connection to the explicit determination of the Jacobian of a cubic curve, and to the description of $SL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \times SL_3(\mathbb{C})$-invariants of tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Furthermore, in Section 4 we deduce from Theorem 3 the explicit combinatorial description of the ring of conjugation invariants of pairs of 3×3 matrices. In particular, we recover the complicated relation due to Nakamoto [21] as a simple consequence of our results on $R(K)$. In summary, the complicated relation mentioned above comes from the simple relation in Theorem 1 by specialization and change of variables.

Let us note finally that R is an instance of a semi-invariant algebra of a quiver, and Theorems 1 and 3 give information on the homogeneous coordinate ring of the moduli space of semistable $(3, 3)$-dimensional representations (cf. [18]) of the generalized Kronecker quiver with three arrows.

2 Characteristic zero

Throughout this section we assume that $K = \mathbb{Q}$, the field of rational numbers. (Everything would hold for any characteristic zero base field.) To simplify notation set $R := R(\mathbb{Q})$, $P := P(\mathbb{Q})$. The homogeneous components of R are polynomial $GL_3(\mathbb{Q})$-modules. Recall that given a representation of $GL_3(\mathbb{Q})$ on some vector space and $\alpha \in \mathbb{Z}^3$ we say that a non-zero vector v is a weight vector of weight α if $\mathrm{diag}(z_1, z_2, z_3) \cdot v = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} v$ for all diagonal elements $\mathrm{diag}(z_1, z_2, z_3) \in GL_3(\mathbb{Q})$. A polynomial $GL_3(\mathbb{Q})$-module is completely reducible, and the isomorphism classes of irreducible polynomial $GL_3(\mathbb{Q})$-modules are labeled by partitions λ with at most three non-zero parts, i.e., $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{N}_0^3$ is a triple of non-negative integers with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Write V_λ for the irreducible polynomial $GL_3(\mathbb{Q})$-module corresponding to λ. Given a polynomial representation

of $GL_3(\mathbb{Q})$, a weight vector is called a *highest weight vector* if it is fixed by all unipotent upper triangular elements in $GL_3(\mathbb{Q})$. Then its weight is necessarily a partition λ, and it generates a $GL_3(\mathbb{Q})$-submodule isomorphic to V_λ.

The $GL_3(\mathbb{Q})$-module structure of R is encoded in its 3-variable Hilbert series

$$H(R;t_1,t_2,t_3) := \sum_{\alpha\in\mathbb{N}_0^3} \dim_{\mathbb{Q}}(R_\alpha) t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \in \mathbb{Z}[[t_1,t_2,t_3]],$$

where R_α denotes the α weight subspace of R. From (1) we know that

$$H(R;t_1,t_2,t_3) = \frac{1+t_1^3t_2^3t_3^3}{(1-t_1^2t_2^2t_3^2)\prod_{i+j+k=3}(1-t_1^it_2^jt_3^k)}.$$

This shows that up to degree ≤ 6, the homogeneous components of R coincide with those of P. Hence P is a $GL_3(\mathbb{Q})$-submodule in R. Denote by P_0 the subalgebra of P generated by the $f_{i,j,k}$. For $g = (g_{ij})_{3\times 3} \in GL_3(\mathbb{Q})$ we have

$$\sum_{i+j+k=3} t_1^it_2^jt_3^k f_{i,j,k}((A_1,A_2,A_3)\cdot g) = \det\left(\sum_{j=1}^3\left(t_j\sum_{i=1}^3 g_{ij}A_i\right)\right) = \det\left(\sum_{i=1}^3\left(\sum_{j=1}^3 g_{ij}t_j\right)A_i\right)$$
$$= \sum_{l+m+n=3} f_{l,m,n}(A_1,A_2,A_3)\left(\sum_{r=1}^3 g_{1r}t_r\right)^l\left(\sum_{r=1}^3 g_{2r}t_r\right)^m\left(\sum_{r=1}^3 g_{3r}t_r\right)^n,$$

hence

$$\sum_{i+j+k=3}(g\cdot f_{i,j,k})t_1^it_2^jt_3^k = \sum_{l+m+n+3} f_{l,m,n}(A_1,A_2,A_3)\left(\sum_{r=1}^3 g_{1r}t_r\right)^l\left(\sum_{r=1}^3 g_{2r}t_r\right)^m\left(\sum_{r=1}^3 g_{3r}t_r\right)^n. \tag{2}$$

So the $f_{i,j,k}$ span a $GL_3(\mathbb{Q})$-submodule, hence P_0 is also a $GL_3(\mathbb{Q})$-submodule, and we see from the Hilbert series that the degree 6 homogeneous component of P_0 has a $GL_3(\mathbb{Q})$-module direct complement in the degree 6 homogeneous component of P isomorphic to $V_{(2,2,2)}$. Taking the multidegree into account we conclude that there exist unique scalars β_i, $i=1,2,3,4$, such that $H := h + \beta_1 f_{2,1,0}f_{0,1,2} + \beta_2 f_{2,0,1}f_{0,2,1} + \beta_3 f_{1,2,0}f_{1,0,2} + \beta_4 f_{1,1,1}^2$ is a highest weight vector (and hence spans the submodule $V_{(2,2,2)}$ mentioned above). To find the values β_i note that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

generate a Zariski dense subgroup in the subgroup of unipotent upper triangular matrices in $GL_3(\mathbb{Q})$. Therefore the condition that H is a highest weight vector is equivalent to the condition that the above two elements of $GL_3(\mathbb{Q})$ fix H. This gives a system of linear equations for the β_i, that can be easily solved (we used CoCoA [8]), and we get that

$$H = h - \frac{1}{3}f_{2,1,0}f_{0,1,2} - \frac{1}{3}f_{2,0,1}f_{0,2,1} + \frac{2}{3}f_{1,2,0}f_{1,0,2} + \frac{1}{12}f_{1,1,1}^2. \tag{3}$$

Denote by P_+ the sum of the positive degree homogeneous components of P. Then by the above considerations we know that P_+R is a $GL_3(\mathbb{Q})$-submodule in R, and the Hilbert series of R shows

that P_+R has a $GL_3(\mathbb{Q})$-module direct complement isomorphic to $V_{(0)} \oplus V_{(3,3,3)}$ (where $V_{(0)}$ is the trivial $GL_3(\mathbb{Q})$-module). Consequently, there is a unique weight vector v in P of weight $(3,3,3)$ (i.e., a multihomogeneous element of multidegree $(3,3,3)$) such that $Q := q + v$ is a highest weight vector (and hence spans the submodule $V_{(3,3,3)}$ mentioned above). Solving a small system of linear equations as in the case of H we obtain

$$\begin{aligned} Q = \quad & q - \frac{1}{2} h f_{1,1,1} + \frac{3}{2} f_{3,0,0} f_{0,3,0} f_{0,0,3} \\ & - \frac{1}{2} f_{3,0,0} f_{0,2,1} f_{0,1,2} - \frac{1}{2} f_{0,3,0} f_{2,0,1} f_{1,0,2} - \frac{1}{2} f_{0,0,3} f_{2,1,0} f_{1,2,0} \\ & - \frac{1}{2} f_{1,1,1} f_{1,2,0} f_{1,0,2} + \frac{1}{2} f_{2,1,0} f_{1,0,2} f_{0,2,1} + \frac{1}{2} f_{1,2,0} f_{2,0,1} f_{0,1,2}. \end{aligned} \tag{4}$$

It follows from (3), (4), (1) that Q, H and the $f_{i,j,k}$ constitute a minimal generating system of R, and that Q satisfies a monic quadratic polynomial with coefficients in P. Note that $Q, H \in R^{SL_3(\mathbb{Q})}$. Next we introduce two other distinguished elements $\widetilde{S}$, $\widetilde{T}$ in $R^{SL_3(\mathbb{Q})}$. Equation (2) shows that the algebraically independent invariants $f_{i,j,k}$ span a $GL_3(\mathbb{Q})$-submodule in R isomorphic to the dual of the space of ternary cubic forms, hence $P_0^{SL_3(\mathbb{Q})}$ is isomorphic to the algebra of $SL_3(\mathbb{Q})$-invariants of ternary cubic forms. The latter was determined in [3], and is generated by two algebraically independent elements S and T. Here S and T are homogeneous polynomials of degree four and six in the coefficients of the general cubic ternary form

$$aX^3 + bY^3 + cZ^3 + 3a_2X^2Y + 3a_3X^2Z + 3b_1XY^2 + 3b_3Y^2Z + 3c_1XZ^2 + 3c_2YZ^2 + 6mXYZ.$$

The expressions S and T can be found in [22], in [9, page 160], or in [2]. Now substitute in S and T the coefficients of the general ternary form by the $f_{i,j,k}$ to get elements $\widetilde{S}$ and $\widetilde{T}$ in P_0. The exact substitution is given by the following table:

a	a_2	a_3	b	b_1	b_3	c	c_1	c_2	m
$f_{3,0,0}$	$\frac{1}{3}f_{2,1,0}$	$\frac{1}{3}f_{2,0,1}$	$f_{0,3,0}$	$\frac{1}{3}f_{1,2,0}$	$\frac{1}{3}f_{0,2,1}$	$f_{0,0,3}$	$\frac{1}{3}f_{1,0,2}$	$\frac{1}{3}f_{0,1,2}$	$\frac{1}{6}f_{1,1,1}$

By (2) this substitution induces a $GL_3(\mathbb{Q})$-module isomorphism from the dual of the space of ternary cubic forms to the subspace of R spanned by the $f_{i,j,k}$. Hence $\widetilde{S}$, $\widetilde{T}$ are $SL_3(\mathbb{C})$-invariants in P_0, and by [3] we have $P_0^{SL_3(\mathbb{C})} = \mathbb{C}[\widetilde{S}, \widetilde{T}]$. Moreover, since H is $SL_3(\mathbb{Q})$-invariant, we have $P^{SL_3(\mathbb{Q})} = (P_0[H])^{SL_3(\mathbb{Q})} = P_0^{SL_3(\mathbb{Q})}[H] = \mathbb{Q}[\widetilde{S}, \widetilde{T}, H]$, a three-variable polynomial algebra. Since Q is also $SL_3(\mathbb{Q})$-invariant, we conclude from $R = P \oplus P \cdot Q$ that

$$R^{SL_3(\mathbb{Q})} = P^{SL_3(\mathbb{Q})} \oplus Q \cdot P^{SL_3(\mathbb{Q})} = \mathbb{Q}[\widetilde{S}, \widetilde{T}, H] \oplus Q \cdot \mathbb{Q}[\widetilde{S}, \widetilde{T}, H].$$

Taking the degrees into account it follows that $Q^2 = \alpha H^3 + \beta H\widetilde{S} + \gamma\widetilde{T}$ for some unique scalars $\alpha, \beta, \gamma \in \mathbb{Q}$. The scalars can be easily found by substituting special matrix triples into the above equality: on skew-symmetric triples all the $f_{i,j,k}$ vanish, hence $\widetilde{T}$ and $\widetilde{S}$ vanish. On the other hand, the value of H on the triple

$$\left(\begin{pmatrix} 0 & -x_1 & -y_1 \\ x_1 & 0 & -z_1 \\ y_1 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_2 & -y_2 \\ x_2 & 0 & -z_2 \\ y_2 & z_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_3 & -y_3 \\ x_3 & 0 & -z_3 \\ y_3 & z_3 & 0 \end{pmatrix} \right)$$

is $\det^2 \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$, whereas the value of Q on this triple is $\det^3 \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$. This

shows that $\alpha = 1$. Note that

$$\det \begin{pmatrix} t_1 & t_3 & 0 \\ 0 & at_1 + t_2 & -bt_1 + t_3 \\ bt_1 + t_3 & 0 & -at_1 + t_2 \end{pmatrix} = t_3^3 + t_2^2 t_1 - b^2 t_1^2 t_3 - a^2 t_1^3$$

(the Weierstrass canonical form of a plane cubic in homogeneous coordinates $(t_1 : t_2 : t_3)$ on $\mathbb{P}^2$). The values of the invariants $\widetilde{S}$, $\widetilde{T}$, H, Q on the corresponding matrix triple

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ b & 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right)$$

are $-b^2/27$, $-4a^2/27$, $-b$, $-a$. It follows that $\beta = 27$ and $\gamma = -27/4$. Hence we proved the following:

Theorem 1 *We have the equality*

$$Q^2 = H^3 + 27H\widetilde{S} - \frac{27}{4}\widetilde{T} \tag{5}$$

where H and Q are given explicitly in (3), (4), and they are characterized (up to non-zero scalar multiples) in R as the unique degree 6 and degree 9 $SL_3(\mathbb{Q})$-invariants in R.

Remark 2 (i) The relation (5) essentially coincides with the relation

$$J^2 = 4\Theta^3 + 108S\Theta H^4 - 27TH^6$$

among the basic covariants of plane cubics (cf. [22]). It was observed by Weil [26] that this gives the equation of the Jacobian of a plane cubic, see [2] for a proof. As it is pointed out in the book of Mukai [20, page 430], an alternative approach to this result can be based on the study of R and its defining relation (5). We mention that the results on the equation of the Jacobian of a plane cubic are extended to arbitrary characteristic (including 2 and 3) in [4]. Our results in Section 3 have relevance for this.

(ii) The algebra $R^{SL_3(\mathbb{Q})}$ can be identified with the algebra of $SL_3(\mathbb{Q}) \times SL_3(\mathbb{Q}) \times SL_3(\mathbb{Q})$-invariants on $\mathbb{Q}^3 \otimes \mathbb{Q}^3 \otimes \mathbb{Q}^3$. The arguments above show that this is a three-variable polynomial algebra generated by $\widetilde{S}$, H, and Q. This result is well known, see [7], [24], [25], [6]. Our results provide an alternative proof, and an alternative interpretation of the basic invariants.

3 Relation over the integers

To shorten the expressions, set

$$f_1 := f_{3,0,0},\; f_2 := f_{2,1,0},\; f_3 := f_{2,0,1},\; f_4 := f_{1,2,0},\; f_5 := f_{1,1,1},$$
$$f_6 := f_{1,0,2},\; f_7 := f_{0,3,0},\; f_8 := f_{0,2,1},\; f_9 := f_{0,1,2},\; f_{10} := f_{0,0,3}.$$

It turns out that $Q^2 - H^3 - 27H\widetilde{S} + \frac{27}{4}\widetilde{T} = A(q, h, f_1, \ldots, f_{10})$, where A is a 12-variable

polynomial with integer coefficients, given explicitly as follows:

$$
\begin{aligned}
&A(q,h,f_1,\ldots,f_{10}) = q^2 - qhf_5\\
&+3qf_1f_7f_{10} - qf_1f_8f_9 - qf_2f_4f_{10} + qf_2f_6f_8 + qf_3f_4f_9 - qf_3f_6f_7 - qf_4f_5f_6\\
&-h^3 + h^2f_2f_9 + h^2f_3f_8 - 2h^2f_4f_6\\
&+3hf_1f_4f_8f_{10} - hf_1f_4f_9^2 - 6hf_1f_5f_7f_{10} + hf_1f_5f_8f_9 + 3hf_1f_6f_7f_9 - hf_1f_6f_8^2\\
&-hf_2^2f_8f_{10} + 3hf_2f_3f_7f_{10} - hf_2f_3f_8f_9 + hf_2f_4f_5f_{10} + hf_2f_4f_6f_9 - hf_2f_6^2f_7\\
&-hf_3^2f_7f_9 - hf_3f_4^2f_{10} + hf_3f_4f_6f_8 + hf_3f_5f_6f_7 - hf_4^2f_6^2 + 9f_1^2f_7^2f_{10}^2\\
&-6f_1^2f_7f_8f_9f_{10} + f_1^2f_7f_9^3 + f_1^2f_8^3f_{10} - 6f_1f_2f_4f_7f_{10}^2 + f_1f_2f_4f_8f_9f_{10}\\
&+3f_1f_2f_5f_7f_9f_{10} - f_1f_2f_5f_8^2f_{10} + 3f_1f_2f_6f_7f_8f_{10} - 2f_1f_2f_6f_7f_9^2\\
&+3f_1f_3f_4f_7f_9f_{10} - 2f_1f_3f_4f_8^2f_{10} + 3f_1f_3f_5f_7f_8f_{10} - f_1f_3f_5f_7f_9^2 - 6f_1f_3f_6f_7^2f_{10}\\
&+f_1f_3f_6f_7f_8f_9 + f_1f_4^3f_{10}^2 - f_1f_4^2f_5f_9f_{10} + f_1f_4^2f_6f_8f_{10} + f_1f_4f_5^2f_8f_{10}\\
&-3f_1f_4f_5f_6f_7f_{10} + f_1f_4f_6^2f_7f_9 - f_1f_5^3f_7f_{10} + f_1f_5^2f_6f_7f_9 - f_1f_5f_6^2f_7f_8 + f_1f_6^3f_7^2\\
&+f_2^3f_7f_{10}^2 - 2f_2^2f_3f_7f_9f_{10} + f_2^2f_3f_8^2f_{10} - f_2^2f_4f_6f_8f_{10} - f_2^2f_5f_6f_7f_{10} + f_2^2f_6^2f_7f_9\\
&-2f_2f_3^2f_7f_8f_{10} + f_2f_3^2f_7f_9^2 - f_2f_3f_4f_5f_8f_{10} + 4f_2f_3f_4f_6f_7f_{10} + f_2f_3f_5^2f_7f_{10}\\
&-f_2f_3f_5f_6f_7f_9 + f_2f_4^2f_5f_6f_{10} - f_2f_4f_6^3f_7 + f_3^3f_7^2f_{10} + f_3^2f_4^2f_8f_{10} - f_3^2f_4f_5f_7f_{10}\\
&-f_3^2f_4f_6f_7f_9 - f_3f_4^3f_6f_{10} + f_3f_4f_5f_6^2f_7.
\end{aligned}
$$

For $i,j,k \in \{1,2,3\}$ denote by $x_{ij}^{(r)}$ the coordinate function on $M_3(\mathbb{Q})^3$ mapping the matrix triple (A_1,A_2,A_3) to the (i,j)-entry of A_r. Then $R(\mathbb{Q})$ contains the subring

$$R(\mathbb{Z}) := R(\mathbb{Q}) \cap \mathbb{Z}[x_{ij}^{(r)} \mid i,j,r = 1,2,3].$$

Theorem 3 *Let K be an infinite field or the ring of integers. Then $R(K)$ is minimally generated as a K-algebra by the twelve elements q,h,f_j, $j = 1,\ldots,10$, satisfying the single algebraic relation $A(q,h,f_1,\ldots,f_{10}) = 0$ (where A is given explicitly above). Moreover, $R(K)$ is a free module with basis $1,q$ over its K-subalgebra generated by the eleven algebraically independent elements $h,f_1,\ldots,f_{10}$.*

Proof. We know already from [10] and Section 2 that the statement holds when K is a field of characteristic zero. We also know already that for any K, the given twelve elements satisfy the relation $A(q,h,f_1,\ldots,f_{10}) = 0$.

Suppose next that K is an infinite field of positive characteristic. We claim that 1 and q generate a free $P(K)$-submodule in $R(K)$. Indeed, otherwise q belongs to the field of fractions of $P(K)$. By the above relation q is integral over $P(K)$. Since $P(K)$ is a unique factorization domain, it follows that q belongs to $P(K)$. Taking the grading of R into account, we conclude that $q = hc + d$, where c is a linear combination of the f_i, and d is a cubic polynomial in the f_i. Now substitute into this equality a triple (A_1,A_2,A_3), where the A_i constitute a basis of the space of 3×3 skew-symmetric matrices. All the f_i vanish on this triple, hence $(hc+d)(A_1,A_2,A_3) = 0$, whereas q does not vanish on this triple as we pointed out in Section 2. So $P(K)\oplus P(K)q \subseteq R(K)$. It follows from the theory of modules with good filtration (cf. [16, page 399]) that the Hilbert series of $R(K)$ coincides with the Hilbert series of $R(\mathbb{Q})$. We know already that the latter coincides with the Hilbert series of $P(K) \oplus P(K)q$, hence we have the equality $R(K) = P(K) \oplus P(K)q$. This shows both the statement on the generators and the relation.

Finally we turn to $R(\mathbb{Z})$. Denote by $P(\mathbb{Z})$ the $\mathbb{Z}$-subalgebra of $R(\mathbb{Z})$ generated by the eleven elements $h,f_1,\ldots,f_{10}$. From the case $K = \mathbb{Q}$ we know that $P(\mathbb{Z})$ is a polynomial ring, and $R(\mathbb{Z})$

contains the free $P(\mathbb{Z})$-submodule $M := P(\mathbb{Z}) \oplus P(\mathbb{Z})q$. Take any $f \in R(\mathbb{Z})$. It follows from the case $K = \mathbb{Q}$ that some positive integer multiple mf of f belongs to M, so $mf = c + dq$, where $c, d \in P(\mathbb{Z})$. We may assume that m is minimal. If $m \neq 1$, then let p be a prime divisor of m, and let L be an infinite field of characteristic p. Reduction mod p of coefficients gives a ring homomorphism $\pi : Z := \mathbb{Z}[x_{ij}^{(r)} \mid i, j, r = 1, 2, 3] \to L[M_3(L)^3]$, and this restricts to a ring homomorphism $\pi : R(\mathbb{Z}) \to R(L)$ and $\pi : P(\mathbb{Z}) \to P(L)$. Since $\pi(mf) = 0$, we get that $\pi(c) + \pi(d)q = 0$ holds in $R(L)$. From the case $K = L$ of our Theorem we know that $1, q$ are independent over $P(L)$, hence $\pi(c) = 0$ and $\pi(d) = 0$, i.e. $c, d \in P(\mathbb{Z}) \cap pZ$. Clearly $P(\mathbb{Z}) \cap pZ = pP(\mathbb{Z})$, since the eleven generators of $P(\mathbb{Z})$ are mapped under π to algebraically independent elements of $R(L)$. Consequently, $(m/p)f \in M$, contradicting the minimality of m. Thus we have proved the equality $R(\mathbb{Z}) = P(\mathbb{Z}) \oplus P(\mathbb{Z})q$. This implies both the statement on the generators of $R(\mathbb{Z})$ and the statement on the relation. $\square$

Remark 4 The fact that a minimal $\mathbb{Z}$-algebra generating system of $R(\mathbb{Z})$ stays a minimal K-algebra generating system of $R(K)$ when exchanging the base ring to any infinite field K is accidental, and the analogous property does not hold in general in similar situations. For example, denote by $R_{n,m}(K)$ the ring of $SL_n \times SL_n$-invariants of m-tuples of $n \times n$ matrices. It is proved in [15] that the method we used to construct generators in the special case $n = m = 3$ (i.e. polarization of the determinant of block matrices) yields in general an (infinite) generating system of $R_{n,m}(K)$ for any infinite base field K, hence also for $K = \mathbb{Z}$. (The latter claim follows in the same way as it is explained by Donkin [16] in a related situation.) However, Proposition 5 below and the results of [14] imply that if m is sufficiently large, then a minimal $\mathbb{Z}$-algebra generating system of $R_{n,m}(\mathbb{Z})$ becomes redundant over fields K whose characteristic is zero or greater than n.

4 Conjugation invariants of pairs of 3×3 matrices

The general linear group $GL_3(K)$ acts on $M_3(K)^2 = M_3(K) \oplus M_3(K)$ by simultaneous conjugation: For $g \in GL_3(K)$ and $A, B \in M_3(K)$ we set $g \cdot (A, B) = (gAg^{-1}, gBg^{-1})$. For any infinite field K denote by $U(K) := K[M_3(K)^2]^{GL_3(K)}$ the corresponding algebra of invariants. Similarly to Section 3, consider

$$U(\mathbb{Z}) := U(\mathbb{Q}) \cap \mathbb{Z}[x_{ij}^{(r)} \mid i, j = 1, 2, 3;\ r = 1, 2]$$

where $x_{ij}^{(r)}$ is the coordinate function assigning to the pair $(A_1, A_2) \in M_3(\mathbb{Q})^2$ the (i, j)-entry of A_r. A minimal system of generators of $U(K)$ was given by Teranishi [23] when $\mathrm{char}(K) = 0$; Nakamoto [21] extended the result for any infinite base field K or $K = \mathbb{Z}$, and determined the single defining relation among the generators. An exact description of $U(K)$ can also be obtained from Theorem 3, using the following statement (proved in [10, Proposition 4.1]):

Proposition 5 *The specialization $x_{ij}^{(3)} \mapsto \delta_j^i$ (where $\delta_j^i = 1$ if $i = j$ and $\delta_j^i = 0$ otherwise) induces a surjection $\varphi : R(K) \to U(K)$.*

Corollary 6 *Let K be an infinite field or the ring of integers. Then $U(K)$ is minimally generated as a K-algebra by the eleven elements $\varphi(q), \varphi(h), \varphi(f_j)$, $j = 1, \ldots, 9$, satisfying the single algebraic relation $A(\varphi(q), \varphi(h), \varphi(f_1), \ldots, \varphi(f_9), 1) = 0$ (where A is given explicitly in Section 3). Moreover, $R(K)$ is a free module with basis $1, \varphi(q)$ over its K-subalgebra generated by the ten algebraically independent elements $\varphi(h), \varphi(f_1), \ldots, \varphi(f_9)$.*

Proof. First we express the φ-images of the generators of $R(K)$ in terms of the usual generators of $U(K)$. Define the functions t, s, d on $M_3(K)$ by the equality

$$\det(zI + A) = z^3 + t(A)z^2 + s(A)z + d(A),$$

where I is the 3×3 identity matrix and $z \in K$ arbitrary. One has the equality

$$s(AB) = t(A^2B^2) + t(AB)t(A)t(B) - t(A^2B)t(B) - t(AB^2)t(A) - s(A)s(B)$$

for $A, B \in M_3(K)$ (see [12, Lemma 2] for a generalization). Furthermore, we have

$$\begin{array}{l} \varphi(f_{3,0,0})(A,B) = d(A),\ \varphi(f_{0,3,0})(A,B) = d(B),\ \varphi(f_{0,0,3})(A,B) = 1, \\ \varphi(f_{2,0,1})(A,B) = s(A), \qquad \varphi(f_{0,2,1})(A,B) = s(B), \\ \varphi(f_{1,0,2}(A,B) = t(A), \qquad \varphi(f_{0,1,2})(A,B) = t(B), \\ \varphi(f_{1,1,1})(A,B) = t(A)t(B) - t(AB), \\ \varphi(f_{2,1,0})(A,B) = t(A^2B) - t(AB)t(A) + s(A)t(B), \\ \varphi(f_{1,2,0})(A,B) = t(AB^2) - t(AB)t(B) + t(A)s(B). \end{array}$$

Applying Amitsur's formula [1] one gets

$$\begin{array}{rcl} \varphi(h)(A,B) & = & -t(A^2B^2) + t(A^2B)t(B) - t^2(A)s(B) + 2s(A)s(B), \\ \varphi(q)(A,B) & = & t(B^2A^2BA) - s(A)s(B)t(AB) - t(A^2B)t(AB)t(B) \\ & & -t(AB^2)t(AB)t(A) + t^2(AB)t(A)t(B). \end{array}$$

Therefore by Theorem 3 and Proposition 5, the eleven elements

$$t(A), s(A), d(A), t(B), s(B), d(B), t(AB), t(A^2B), t(AB^2), t(A^2B^2), t(B^2A^2BA) \tag{6}$$

generate $U(K)$. Moreover, $t(B^2A^2BA)$ satisfies a monic quadratic relation over the subalgebra $W(K)$ of $U(K)$ generated by the first ten elements. Since by general principles on group actions, the transcendence degree of $U(K)$ is ten, the first ten generators are algebraically independent. Moreover, $t(B^2A^2BA)$ does not vanish on the pair $(E_{21} - E_{32}, E_{12} + E_{23})$ (where E_{ij} is the matrix unit whose only non-zero entry is a 1 in the (i, j)-position), whereas all the first nine generators vanish on this pair. Since the tenth generator has degree 4 and the eleventh generator has degree 6, it follows that $t(B^2A^2BA)$ is not contained in $W(K)$. Hence by the integral closeness of $W(K)$ and by the quadratic relation we conclude that $U(K) = W(K) \oplus t(B^2A^2BA)W(K)$. The statements in our Corollary obviously follow. $\square$

Corollary 7 *When K is an infinite field with $\mathrm{char}(K) \neq 2$ or 3, then the algebra $U(K)$ is minimally generated by $\varphi(Q), \varphi(H), \varphi(f_1), \ldots, \varphi(f_9)$, and these generators satisfy the single algebraic relation*

$$\varphi(Q)^2 = \varphi(H)^3 + 27\varphi(H)\varphi(\widetilde{S}) - \frac{27}{4}\varphi(\widetilde{T}).$$

Remark 8 (i) Expressing the left-hand side of $A(\varphi(q), \varphi(h), \varphi(f_1), \ldots, \varphi(f_9), 1) = 0$ in terms of the generators (6) we obtain a transparent derivation of the relation found originally by hard computational labour by Nakamoto [21]. We include this relation in the Appendix.

(ii) The form of the relation in Corollary 7 is rather simple (or better to say that the complication is built into the nineteenth century expressions for S and T due to [3]): Indeed, the quartic or sextic generators appear only in three terms, and the remaining 9 generators appear only in two prominent classically known (though complicated) expressions.

(iii) We note that working over a characteristic zero base field, another minimal generating system of $U(K)$ is found in [5], such that the relation between them takes a simpler form than the relation in [21]. This relation is of different nature than the one in Corollary 7.

Appendix. Setting

$$
\begin{array}{cccccc}
t_1 := t(A), & s_1 := s(A), & d_1 := d(A), & t_2 := t(B), & s_2 := s(B), & d_2 := d(B) \\
r := t(B^2A^2BA), & k := t(A^2B^2), & w_1 := t(A^2B), & w_2 := t(AB^2), & z := t(AB) &
\end{array}
$$

we have

$$
\begin{aligned}
0 \; = \; & r^2 - rkz + rkt_1t_2 - rw_1w_2 - rw_1t_1t_2^2 - rw_2t_1^2t_2 \\
+ \; & rzt_1^2t_2^2 + 3rd_1d_2 - rd_1s_2t_2 - rd_2s_1t_1 - rs_1s_2t_1t_2 \\
+ \; & k^3 - 2k^2w_1t_2 - 2k^2w_2t_1 + k^2zt_1t_2 - 5k^2s_1s_2 + k^2s_1t_2^2 + k^2s_2t_1^2 \\
+ \; & kw_1^2s_2 + kw_1^2t_2^2 + kw_1w_2z + 2kw_1w_2t_1t_2 - kw_1zs_2t_1 - kw_1zt_1t_2^2 - 3kw_1d_2s_1 \\
+ \; & kw_1d_2t_1^2 + 9kw_1s_1s_2t_2 - 2kw_1s_1t_2^3 - 2kw_1s_2t_1^2t_2 + kw_2^2s_1 + kw_2^2t_1^2 - kw_2zs_1t_2 \\
- \; & kw_2zt_1^2t_2 - 3kw_2d_1s_2 + kw_2d_1t_2^2 + 9kw_2s_1s_2t_1 - 2kw_2s_1t_1t_2^2 - 2kw_2s_2t_1^3 + kz^2s_1s_2 \\
- \; & 6kzd_1d_2 + 4kzd_1s_2t_2 - kzd_1t_2^3 + 4kzd_2s_1t_1 - kzd_2t_1^3 - 8kzs_1s_2t_1t_2 + 2kzs_1t_1t_2^3 \\
+ \; & 2kzs_2t_1^3t_2 + 3kd_1d_2t_1t_2 - 2kd_1s_2^2t_1 - 2kd_2s_1^2t_2 + 8ks_1^2s_2^2 - 2ks_1^2s_2t_2^2 - 2ks_1s_2^2t_1^2 \\
+ \; & w_1^3d_2 - w_1^3s_2t_2 - w_1^2w_2s_2t_1 - 2w_1^2zd_2t_1 + 2w_1^2zs_2t_1t_2 + 4w_1^2d_2s_1t_2 - w_1^2d_2t_1^2t_2 \\
- \; & w_1^2s_1s_2^2 - 4w_1^2s_1s_2t_2^2 + w_1^2s_1t_2^4 + w_1^2s_2t_1^2t_2^2 - w_1w_2^2s_1t_2 + w_1w_2zs_1t_2^2 + w_1w_2zs_2t_1^2 \\
- \; & 6w_1w_2d_1d_2 + 4w_1w_2d_1s_2t_2 - w_1w_2d_1t_2^3 + 4w_1w_2d_2s_1t_1 - w_1w_2d_2t_1^3 - 8w_1w_2s_1s_2t_1t_2 \\
+ \; & 2w_1w_2s_1t_1t_2^3 + 2w_1w_2s_2t_1^3t_2 + w_1z^2d_2s_1 + w_1z^2d_2t_1^2 - w_1z^2s_1s_2t_2 - w_1z^2s_2t_1^2t_2 \\
+ \; & 6w_1zd_1d_2t_2 + w_1zd_1s_2^2 - 4w_1zd_1s_2t_2^2 + w_1zd_1t_2^4 - 8w_1zd_2s_1t_1t_2 + 2w_1zd_2t_1^3t_2 \\
+ \; & w_1zs_1s_2^2t_1 + 8w_1zs_1s_2t_1t_2^2 - 2w_1zs_1t_1t_2^4 - 2w_1zs_2t_1^3t_2^2 - 3w_1d_1d_2s_2t_1 - 2w_1d_1d_2t_1t_2^2 \\
+ \; & 2w_1d_1s_2^2t_1t_2 + 4w_1d_2s_1^2s_2 + 2w_1d_2s_1^2t_2^2 - w_1d_2s_1s_2t_1^2 - 8w_1s_1^2s_2^2t_2 + 2w_1s_1^2s_2t_2^3 \\
+ \; & 2w_1s_1s_2^2t_1^2t_2 + w_2^3d_1 - w_2^3s_1t_1 - 2w_2^2zd_1t_2 + 2w_2^2zs_1t_1t_2 + 4w_2^2d_1s_2t_1 - w_2^2d_1t_1t_2^2 \\
- \; & w_2^2s_1^2s_2 - 4w_2^2s_1s_2t_1^2 + w_2^2s_1t_1^2t_2^2 + w_2^2s_2t_1^4 + w_2z^2d_1s_2 + w_2z^2d_1t_2^2 - w_2z^2s_1s_2t_1 \\
- \; & w_2z^2s_1t_1t_2^2 + 6w_2zd_1d_2t_1 - 8w_2zd_1s_2t_1t_2 + 2w_2zd_1t_1t_2^3 + w_2zd_2s_1^2 - 4w_2zd_2s_1t_1^2 \\
+ \; & w_2zd_2t_1^4 + w_2zs_1^2s_2t_2 + 8w_2zs_1s_2t_1^2t_2 - 2w_2zs_1t_1^2t_2^3 - 2w_2zs_2t_1^4t_2 - 3w_2d_1d_2s_1t_2 \\
- \; & 2w_2d_1d_2t_1^2t_2 + 4w_2d_1s_1s_2^2 - w_2d_1s_1s_2t_2^2 + 2w_2d_1s_2^2t_1^2 + 2w_2d_2s_1^2t_1t_2 - 8w_2s_1^2s_2^2t_1 \\
+ \; & 2w_2s_1^2s_2t_1t_2^2 + 2w_2s_1s_2^2t_1^3 + z^3d_1d_2 - z^3d_1s_2t_2 - z^3d_2s_1t_1 + z^3s_1s_2t_1t_2 - 5z^2d_1d_2t_1t_2 \\
+ \; & 4z^2d_1s_2t_1t_2^2 - z^2d_1t_1t_2^4 + 4z^2d_2s_1t_1^2t_2 - z^2d_2t_1^4t_2 - z^2s_1^2s_2^2 - 4z^2s_1s_2t_1^2t_2^2 + z^2s_1t_1^2t_2^4 \\
+ \; & z^2s_2t_1^4t_2^2 + 6zd_1d_2s_1s_2 + zd_1d_2s_1t_2^2 + zd_1d_2s_2t_1^2 + 2zd_1d_2t_1^2t_2^2 - 4zd_1s_1s_2^2t_2 \\
+ \; & zd_1s_1s_2t_2^3 - 2zd_1s_2^2t_1^2t_2 - 4zd_2s_1^2s_2t_1 - 2zd_2s_1^2t_1t_2^2 + zd_2s_1s_2t_1^3 + 8zs_1^2s_2^2t_1t_2 \\
- \; & 2zs_1^2s_2t_1t_2^3 - 2zs_1s_2^2t_1^3t_2 + 9d_1^2d_2^2 - 6d_1^2d_2s_2t_2 + d_1^2d_2t_2^3 + d_1^2s_2^3 - 6d_1d_2^2s_1t_1 + d_1d_2^2t_1^3 \\
- \; & 2d_1d_2s_1s_2t_1t_2 + 2d_1s_1s_2^3t_1 + d_2^2s_1^3 + 2d_2s_1^3s_2t_2 - 4s_1^3s_2^3 + s_1^3s_2^2t_2^2 + s_1^2s_2^3t_1^2.
\end{aligned}
$$

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